XXIV ONPCE: Oficina Nacional de Problemas de Corte e Empacotamento, Planejamento e Programação da Produção e Correlatos

On von Nemann's cellular automata on grids

Nei Yoshihiro Soma[1]
DCTA/ITA/IEC (PGEEC-I, PPG-PO)

José Prado de Melo
(In Memoriam)

Cellular Automata (CA) are machines possessing a discrete model of computation. They consist on a collection of *cells* disposable in a regular and homogenous grid. These machines appeared in a work of John von Neumann and Stanislaw Ulam in an epoch that the computer had yet to be invented. After John Conway's Game of Life in the 1970's the interested for its study blossomed [4]. The research on CA's gained further momentum with the extensive studies of Stephen Wolfram since the publication of his "Cellular automata as models of complexity" that appeared in Nature (and front-cover) [5].

This work focus in a special type of CA and brings new approaches for dealing with an *open problem* already established by von Neumann, that is, to find a general condition for irreversibility of a class of two-dimensional cellular automata on square grids (σ^+-automata). We give here some new proofs to the uni and bidimensional cases, with von Neumann´s problem remaining elusive.

The σ^+-automata is defined as a matrix $n \times n$ of cells over $GF(2)$. Time evolves in steps as a syncronous and discrete evolution of those cells. The neighbourhood of any cell c comes from a pre-defined set of cells and those influence the state of c in the next instant of time. The von Neumann two-dimensional CA, from now on referred as σ_2^+. Next definitions are *transcriptions* (with typo corrections) of an early work of the authors [1].

$$c_{i,j}^{t+1} = c_{i-1,j}^t + c_{i,j-1}^t + c_{i,j}^t + c_{i+1,j}^t + c_{i,j+1}^t \ (\mathrm{mod}\ 2),$$

being $c_{i,j}^t$ the state of cell i,j at time n, $i, j \in \{1, \cdots, n\}$ and $c_{i,j}^t = 0$ or 1. The *states* of the *cells* $c_{0,i}$, $c_{i,0}$, $c_{i,n+1}$ and $c_{n+1,i}$ are always 0 for any time t. Any configuration of a CA can be represented by a system of linear equations over $GF(2)$ of the form $B.x^t = x^{t+1}$, where x^t and x^{t+1} are elements amidsts a finite set with 2^{n^2} different *configurations*, related to instants t and $t + 1$ and B is the adjacency matrix associated to the $n \times n$ grid graph. End of *transcription*.

The unidimensional case, σ_1^+, provides some useful insights to the intrinsically recursive structure of the problem and we briefly mention it. The σ_1^+ is defined to a tape containing n cells and its adjacency matrix is a $n \times n$ Jacobi matrix with only 0´s and 1´s, let us call it $B_n^{[1]}$. To σ_1^2 the associated matrix is a block tridiagonal containing the identity matrix of order $n \times n$ bellow and above the main diagonal that has $B_n^{[1]}$ as its blocks.

To detect irreversibility to either σ_1^+ or σ_2^+ it sufficives determining the determinant to those adjacency matrices. Sutner [3] showed that σ_1^+ with n cells is irrevesible if and only if

[1] nys@ita.br

$n \equiv 2 \pmod 3$ by using the detection of cycles in graphs. Also, in [2] obtained the same result by calculating the characteristic polynomial roots of σ_1^+ matrix.

There is a much easier way however: $|B_n^{[1]}| = |B_{n-1}^{[1]}| - |B_{n-2}^{[1]}|$. Addionally, it is clear that $|B_1^{[1]}| = 1$ and $|B_2^{[1]}| = 0$. With these, we obtain the following sequence: $|B_3^{[1]}| = |B_4^{[1]}| = -1$, $|B_1^{[5]}| = 0$, $|B_6^{[1]}| = |B_7^{[1]}| = 1, \cdots$. Moreover, from them, it is not difficult to get, as in [2]:

$$\left| B_n^{[1]} \right| = \frac{2\sqrt{3}}{3} \cos\left(\frac{n\pi}{3} - \frac{\pi}{6}\right), \quad n = 1, 2, 3, \cdots . \tag{1}$$

Matrix $B_n^{[2]}$ associated with σ_2^+ has a similar derivation for its determinant but since it is a Jacobi tridiagonal in blocks it now is a polynomial in $B_n^{[1]}$:

$$\left| B_n^{[2]} \right| = | \, I_n \, |^{n-1} . |\Pi_n|,$$

where I_n is the identity matrix of order n and Π_n is a polynomial in $B_n^{[1]}$ given by the following recurrence in $GF(2)$: $\Pi_0 = O_n$, $\Pi_1 = B_n^{[1]}$ and for $k = 2, \cdots, n$; $\Pi_n = B_n^{[1]}.\Pi_{n-1} + \Pi_{n-2}$. For instance, $\Pi_2 = (B_2^{[1]})^2 + I_2$ and $\Pi_3 = (B_2^{[1]})^3$. Notice that these polynomials have different matrices orders, Π_2 and Π_3, respectively, have are of size 2×2 and 3×3. These polynomials should not be computed recursively, since it would require $\Omega(1.5^n)$. Even by Dynamic Programming using memoization is costly since there would require indices manipulation to reduce to $GF(2)$. We will prove (and present) an optimal algorithm for determining them.

Dedication: This paper is in loving memory of one of the two authors, José Prado de Melo (1948–2020); Fábio Carneiro Mokarzel (1951–2021) and Waldecir João Perrella (1949–2023).

Referências

[1] N.Y.Soma and J.P. Melo. Cut of squares with von Neumann neighbourhood: some sufficient conditions, in: C.E. Ferreira, F.K. Miyazawa, Y. Wakabayashi (Eds.), I Oficina Nacional de Corte Empacotamento e Correlatos, University of São Paulo, São Paulo, 37—42, (in Portuguese), 1996.

[2] N.Y. Soma and J.P. Melo. On irreversibility of von Neumann additive cellular automata on grids *Discrete Applied Mathematics*. 154:861–866, 2006.

[3] K. Sutner. Linear Cellular Automata and the Garden-of-Eden *The Mathematical Intelligencer*, 11(**2**): 49–53, 1989.

[4] Wikipedia, verbete *"Cellular Automata"*, last seen on 29, January, 2023.

[5] S. Wolfram. Cellular Automata as models of complexity *Nature*. 311(**5985**): 419–424, 4 October 1984.